

# Chern-Simons theory encoded on a spin chain

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We show how the observables of  $U(N)$  Chern-Simons theory on  $S^3$  can be reproduced with the thermal correlation functions of a 1d spin chain of the XX type, with a suitable extension characterized by exponentially decaying interactions between infinitely many neighbours. We show that for this model, the correlation functions of the spin chain at a finite temperature  $\beta = 1$  give the partition function, quantum dimensions and the  $S$ -matrix.

## INTRODUCTION

The study of topological field theories has been a subject of continued interest in the last decades (see [1] for an early review). Chief among topological field theories is Chern-Simons theory due to its relevance in three dimensional topology and knot theory [2], its role as the effective field theory of the fractional quantum Hall effect [3] and, more recently, its appearance also in the study of topological quantum computation [4–6] and in the description of topological strings [7].

In recent years, motivated by the realm of quantum computation, a number of proposals have been devised to simulate different quantum field theories (QFT), including non Abelian gauge theories [8]–[15]. In a somehow related but different direction, in [16], we introduced a connection, illustrated for the case of low-energy QCD, between some QFT and 1D spin systems, based on the existence of a random matrix description. This allowed us to relate crucial properties of the QFT with physically meaningful properties of the 1D system.

In this paper, following that path, we introduce a new spin chain Hamiltonian, expected to be in the same universality class as the XX-Hamiltonian, whose thermal correlation functions correspond exactly with the main observables of Chern-Simons theory. We will discuss how a finite chain approximates these observables with an exponentially small error in the size of the chain and how finite chains models may describe Chern-Simons theory coupled with matter in the fundamental representation. We will also explain how the interactions of the chain can be modified while preserving its topological properties and show that the presence of a magnetic field in the Hamiltonian only affects the overall normalization of the Chern-Simons theory observables.

## RESULTS

### Chern-Simons theory

Let us first remind the basics of Chern-Simons theory, which is a three dimensional gauge theory with a simply connected and compact non-Abelian Lie group  $G$  and the

Chern-Simons action, given by [2]

$$S_{\text{CS}}(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (1)$$

where  $\text{Tr}$  is the trace in the fundamental representation,  $A$  is the connection, a 1-form valued on the corresponding Lie algebra, and  $k \in \mathbb{Z}$  is the level. The manifold  $M$  is a three dimensional compact manifold which, in this work, will be chosen to be  $S^3$ . The  $q$ -parameter of Chern-Simons theory is defined in terms of the level  $k$  by  $q = \exp(2\pi i/(k+h))$  where  $h$  is the dual Coxeter number of the Lie algebra, which is just  $h = N$  in this paper since we choose  $G = U(N)$ . In the random matrix theory description of Chern-Simons theory that we will employ,  $q$  is treated as a real number  $q = \exp(-\gamma)$  (with  $\gamma > 0$ ). This treatment directly reproduces the results in topological string theory [7] and  $q$ -deformed of 2d Yang-Mills theory [17]. The usual Chern-Simons expressions are recovered at the end by making the identification  $\gamma = 2\pi i/(k+h)$  [18]. The original study of knot polynomial invariants by Jones also considers  $q$  real [19] although it is possible to extend it to the case of roots of unity, in which case the necessary representation theory of the Hecke algebra is more delicate to deal with. In this paper we focus on  $q$  real.

Let us introduce now a succinct reminder of the solution of Chern-Simons theory. In [2], a concrete description of the (finite-dimensional) Hilbert space of the theory was given: it is the space of conformal blocks of a Wess-Zumino-Witten (WZW) model [20] on  $\Sigma$  with gauge group  $G$  and level  $k$ . Here  $\Sigma$  denotes the Riemann surface which is the common boundary in the Heegard splitting of the manifold  $M$  [21]. In the case studied here,  $M = S^3$ , the Riemann surface is a torus and for  $\Sigma = T^2$ , the space of conformal blocks is in one to one correspondence with the integrable representations of the affine Lie algebra associated to  $G$  at level  $k$  [2, 7]. The states in the Hilbert state of the torus  $\mathcal{H}(T^2)$ , which can be chosen to be orthogonal, are denoted by  $|\lambda\rangle$ , where  $\lambda$  is the representation associated to the integrable representation [7]. In addition, there is a special class of homeomorphisms of  $T^2$  that have a simple expression as operators in  $\mathcal{H}(T^2)$ : the  $S$  and  $T$  transformations which are the generators of the  $SL(2, \mathbb{Z})$  group. The matrix elements of these trans-

formations,  $T_{pp'}$  and  $S_{pp'}$ , have a very explicit expression in the basis of the integrable representations, but these will not be needed here. What is relevant here is that this non-perturbative exact solution of Chern-Simons theory computes the observables of the theory, giving [2, 7]

$$Z(S^3) = \langle 0|S|0 \rangle = S_{00}.$$

This is generalized to obtain the result of the path integral in  $S^3$  with the insertion of knots and links. Considering a solid torus where a Wilson line in representation  $\lambda$  has been inserted (by inserting a Wilson loop  $\mathcal{O}_\lambda = \text{Tr}_\lambda U$  in the representation  $\lambda$  in the path integral), then the corresponding state is  $|\lambda\rangle$ . Gluing this to an empty solid torus after an S-transformation, one obtains the unknot on  $S^3$

$$Z(S^3, \mathcal{O}_\lambda) = \langle 0|S|\lambda \rangle = S_{0\lambda}. \quad (2)$$

The normalization of (2) is the quantum dimension of the representation

$$\dim_q \lambda = \frac{\langle 0|S|\lambda \rangle}{\langle 0|S|0 \rangle} = \frac{S_{0\lambda}}{S_{00}}.$$

The link formed by two unknots in representations  $\lambda$  and  $\mu$  is the Hopf link, given by

$$W_{\mu\lambda} = \frac{\langle \mu|S|\lambda \rangle}{\langle 0|S|0 \rangle} = \frac{S_{\mu\lambda}}{S_{00}}.$$

These observables are (quantum) topological invariants of the manifold [22] and have an important interpretation in terms of the properties (fusion and braiding) of the quasiparticles of a topological quantum field theory. In particular, the elements of the S-matrix are closely related to quasiparticle braiding since  $S_{ab}$  is equal to the amplitude for creating  $a\bar{a}$  and  $b\bar{b}$  pairs, braiding  $a$  and  $b$ , and annihilating again in pair. In addition, as shown by Verlinde's formula, the S-matrix not only contains information about braiding, but also about fusion [4]. The quantum dimension of an anyonic species is a measure for the effective number of degrees of freedom of the internal Hilbert space of the corresponding particle type [23].

### Main result

The main result of this work is to show that all these elements  $S_{\lambda\mu}$  can be recovered as thermal (with  $\beta = 1$ ) correlation functions of a new 1D Hamiltonian with exponentially decaying interactions, namely:

**Main result.**

$$S_{\lambda\mu} = \langle \uparrow | \sigma_{j_1}^+ \cdots \sigma_{j_N}^+ e^{-\hat{H}_{CS}} \sigma_{l_1}^- \cdots \sigma_{l_N}^- | \uparrow \rangle \quad (3)$$

where

$$\begin{aligned} \hat{H}_{CS} = & - \sum_{i,n \in \mathbb{Z}_+} \frac{1}{2n \sinh(\frac{n\gamma}{2})} (\sigma_i^- \otimes \sigma_{i+n}^+ + \sigma_i^- \otimes \sigma_{i-n}^+) \\ & + \frac{h}{2} \sum_i (\sigma_i^z - \mathbb{1}), \end{aligned} \quad (4)$$

$S_{\lambda,\mu}$  are the entries of the S-matrix of Chern-Simons theory on  $S^3$  and gauge group  $U(N)$ ,  $|\uparrow\rangle$  denotes a ferromagnetic state, characterized by having all of its spins up  $|\uparrow\rangle = \otimes_i |\uparrow\rangle$  and  $q = e^{-\gamma}$ . The representations  $\lambda$  and  $\mu$  are described by Young tableaux [7] and are related to the pattern of flipped spins by  $\lambda_r = j_r - N + r$  and  $\mu_r = l_r - N + r$  [16]. There is freedom in the choice of the magnetic field  $h$ , which only affects the overall normalization factor in the thermal correlation functions. We shall discuss its role in the context of the framing of the Chern-Simons theory.

Note that the rank of the gauge group coincides with the number of flipped spins in the thermal correlation function of the spin chain. As particular cases we get the partition function and the quantum dimensions in Chern-Simons theory:

$$\begin{aligned} S_{00} &= \langle \dots, \underbrace{\uparrow, \downarrow, \dots, \downarrow}_N | e^{-\hat{H}_{CS}} | \underbrace{\downarrow, \dots, \downarrow, \uparrow}_N, \dots \rangle, \\ S_{0\lambda} &= \langle \uparrow | \sigma_{j_1}^+ \cdots \sigma_{j_N}^+ e^{-\hat{H}_{CS}} | \underbrace{\downarrow, \dots, \downarrow}_N, \uparrow, \dots \rangle. \end{aligned}$$

Exactly as in [16], where we connected low-energy QCD at finite volume with the XX-Hamiltonian, the key to obtain this result lies on getting a random matrix model expression for both sides of equation (3).

To do that we consider the XX Hamiltonian extended to admit generic interactions, denoted by  $a_j$ , to infinitely many neighbors

$$\hat{H}_{\text{Gen}} = - \sum_i \sum_{j \in \mathbb{Z}} a_j (\sigma_i^- \otimes \sigma_{i+j}^+) + \frac{h}{2} \sum_i (\sigma_i^z - \mathbb{1}). \quad (5)$$

We have the following commutation relations

$$[\sigma_j^+, \hat{H}] = -\sigma_j^z \sum_{i \in \mathbb{Z}} a_i \sigma_{j+i}^+ \quad (6)$$

which, reasoning as in [16], give for the  $N = 1$  correlation function  $G_{jl}(\beta) = \langle \uparrow | \sigma_j^+ e^{-\beta \hat{H}_{\text{Gen}}} \sigma_l^- | \uparrow \rangle$  the following differential-difference equation:

$$\frac{d}{d\beta} G_{jl}(\beta) = \sum_{i \in \mathbb{Z}} a_i G_{j+i,l}(\beta). \quad (7)$$

The generating function associated to these correlators  $g_\beta(\lambda) = \sum_{j=-\infty}^{\infty} G_{jl}(\beta) \lambda^j$  reads

$$g_\beta(\lambda) = g_0(\lambda) \exp \left( \beta \sum_{i \in \mathbb{Z}} a_i \lambda^i \right). \quad (8)$$

Now, as in [16] (and previously in [24, 25] for the nearest-neighbor case), one obtains the general thermal correlation function  $\langle \uparrow | \sigma_{j_1}^+ \dots \sigma_{j_N}^+ e^{-\beta \hat{H}_{Gen}} \sigma_{l_1}^- \dots \sigma_{l_N}^- | \uparrow \rangle$  as a Toeplitz minor whose entries are the  $N = 1$  solution, which is equivalent to a random matrix ensemble with the insertion of two Schur polynomials [26]. That is, we get

$$\begin{aligned} & \langle \uparrow | \sigma_{j_1}^+ \dots \sigma_{j_N}^+ e^{-\beta \hat{H}_{Gen}} \sigma_{l_1}^- \dots \sigma_{l_N}^- | \uparrow \rangle = \\ & \frac{e^{\beta h N}}{(2\pi)^N n!} \int_{-\pi}^{\pi} d\varphi_1 \dots \int_{-\pi}^{\pi} d\varphi_N \prod_{1 \leq j < k \leq N} |e^{i\varphi_k} - e^{i\varphi_j}|^2 \quad (9) \\ & \times \left( \prod_{j=1}^N g_{\beta}(\varphi_j) \right) \frac{\hat{s}_{\mu}(e^{i\varphi_1}, \dots, e^{i\varphi_N}) \hat{s}_{\lambda}(e^{i\varphi_1}, \dots, e^{i\varphi_N})}{\hat{s}_{\mu}(e^{i\varphi_1}, \dots, e^{i\varphi_N}) \hat{s}_{\lambda}(e^{i\varphi_1}, \dots, e^{i\varphi_N})} \end{aligned}$$

where  $\hat{s}_{\lambda}(e^{i\varphi_1}, \dots, e^{i\varphi_N})$  is a Schur polynomial and the representations  $\lambda$  and  $\mu$  are indexed, respectively, by the partitions  $\{\lambda_i\}_{i=1}^N$  and  $\{\mu_i\}_{i=1}^N$ , whose specific relationship with the sequences  $\{j_i\}_{i=1}^N$  and  $\{l_i\}_{i=1}^N$  in the spin operators was given above, just after Eq. (4) [24].

The next step is to recall from [27, 28] that  $U(N)$  Chern-Simons theory on  $S^3$  can be described in terms of a unitary matrix model. In particular, the matrix model is of the type (9) but with a weight function

$$\Theta_3(e^{i\varphi}|q) = \sum_{n=-\infty}^{\infty} q^{n^2/2} e^{in\varphi}, \quad (10)$$

which is Jacobi's third theta function.

Hence, to reproduce Chern-Simons theory we have to consider the generalized spin chain (5) in the specific setting where the resulting generating function (8) turns out to be the theta function (10). The Fourier coefficients of  $\ln \Theta_3(e^{i\varphi}|q)$  can be easily obtained by using the product form of (10) (the Jacobi triple product identity), giving

$$a_n = \frac{1}{2n \sinh\left(\frac{n\gamma}{2}\right)}, \quad (11)$$

with  $n$  denoting positive and negative integers and with  $q = \exp(-\gamma)$  the  $q$ -parameter of Chern-Simons theory. In addition, we have to choose  $\beta = 1$  in (8) because plugging (11) in (8) gives  $\Theta_3^{\beta}(e^{i\varphi}|q)$ . The orthogonality of the states fixes the initial condition  $G_{j,l}(\beta = 0) = \delta_{j,l}$  and this implies that  $g_0(\lambda) = 1$ . This finishes the proof of the main result.

Notice that the Fourier coefficient of order zero, which is  $a_0 = N \sum_{n=1}^{\infty} \ln(1 - q^n)$  has not been included. It only contributes with a normalization factor and hence can be taken into account by fixing the value of the external magnetic field. In any case, this choice of normalization gives an exact correspondence with the unitary matrix model with a theta function [28, 29] whose partition function is explicitly given below where we also explain that framing can be taken into account by a proper choice of the magnetic field.

## Finite chain corrections

The results so far have been exact properties of the spin chain correlators for an infinite spin chain. In the case of a more realistic finite chain of  $L$  sites, the solution to the equation (7) is discrete [25].

Therefore, since the solution of (7) is the weight function of the matrix model, the model is consequently discretized, in which case the thermal average is nothing but the Riemann sum associated to the integral (9), when we evaluate on the vertices of a lattice division of the hypercube  $[-\pi, \pi]^N$  of length  $\frac{2\pi}{L}$ . A recent result [30] shows that the error obtained by this particular Riemann sum approximation decreases exponentially with  $L$ . More concretely, with  $c$  a positive constant, the relative error is  $\mathcal{O}(e^{-c(L-N)})$  as  $L - N \rightarrow \infty$ , even if  $N$  also goes to  $\infty$ . This result holds for Toeplitz determinants of discrete measure under very general conditions and it immediately applies to  $S_{00}$ . The same result is expected to hold for  $S_{\mu\lambda}$  because due to (3) and (9) we know it is the determinant of the minor of a Toeplitz matrix [26]. These results imply that the quantum topology of a manifold can be probed with a finite ring of spins interacting as described [40].

The finite case is also of special interest in its own right because the discretization of the matrix model also emerges naturally in the study of Chern-Simons theory on  $S^2 \times S^1$  with matter in the fundamental representation of the gauge group (vector matter). In recent work [31], it has been shown that, in the large  $N$  limit, the observables of that theory are given by a discrete unitary matrix model with a certain potential  $V(U) = T^2 V_2 v(U)$ , where  $T$  is the temperature (inverse of the radius of  $S^1$ ),  $V_2$  is the volume of  $S^2$  and  $v(U)$  is the potential that has to be computed, in an effective field theory approach, for every theory (that is, depends on the choice of matter made) [31].

Thus, the matrix model description in [31] is precisely the one that holds for, more realistic, *finite* spin chains. Notice that the potential now is not  $\ln \theta_3(e^{i\varphi}, q)$  in general. This simply means that other choices for the generic coefficients  $a_j$  should be made, but, as we have seen, any system characterized by a well-defined unitary matrix model with single trace terms  $c_n \text{Tr} U^n$  with  $n \in \mathbb{Z}$  in the potential is described in the same manner by the spin chain model (5) with interactions given by  $a_n = c_n$ .

## Freedom in the interactions and the magnetic field

It is also interesting to explore how robust are the quantum topological properties of the spin chain. Namely, is the interaction chosen in (4) unique? In particular, for the Chern-Simons unitary ensemble, it is known that  $g(\varphi) = \theta_3^{-1}(-e^{i\varphi}, q)$  is also a valid

weight function for the matrix model to describe Chern-Simons theory [32]. In our context, this implies that we can modify the interactions in (4) with an alternating ferromagnetic-antiferromagnetic coupling. If one considers, as above, the Fourier coefficients of  $\ln g(\varphi) = -\ln \theta_3(-e^{i\varphi}, q)$ , then the corresponding Fourier coefficients (11) are modified accordingly to

$$b_n = \frac{(-1)^{n+1}}{2n \sinh\left(\frac{n\gamma}{2}\right)}, \quad (12)$$

and the resulting spin chain

$$\begin{aligned} \hat{H}_{CS} = & - \sum_{i, n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{2n \sinh\left(\frac{n\gamma}{2}\right)} (\sigma_i^- \otimes \sigma_{i+n}^+ + \sigma_i^- \otimes \sigma_{i-n}^+) \\ & + \frac{h}{2} \sum_i (\sigma_i^z - \mathbb{1}), \end{aligned} \quad (13)$$

possesses the same topological properties. Remarkably, we are lead now to an alternating ferromagnetic-antiferromagnetic interaction with the same decay properties. However, the first and leading interaction term is ferromagnetic, as above. This equivalence of spin chain models is exact for the partition function  $S_{00}$  [32], which is given by a matrix model without any insertion of a Schur polynomial. Regarding the quantum dimensions or the full topological S-matrix, the alternative choice (12) gives the same observables but with transposed partitions  $\lambda', \mu'$  (the columns of the Young tableaux becomes rows and conversely). Thus, alternatively, if one wants equality of the correlators the respective pattern of flipped spins is different (but immediately related) depending on the election of (11) or (12).

Notice that there is also freedom in the choice of the external magnetic field since Chern-Simons theory is actually a theory of *framed* knots and links. Framing  $\Pi$  on a three-manifold  $M$  (here  $M = S^3$ ), is a choice of trivialization of the bundle  $TM \oplus TM$  [33]. It contributes multiplicatively to the observables and, for gauge group  $G$  its explicit expression is [29]

$$\delta(M, \Pi) = \exp\left(\frac{2\pi i s}{24} c\right) = \exp\left(\frac{\pi i s(N^2 - 1)k}{(k + N)}\right), \quad (14)$$

where  $c$  is the central charge of the associated Wess-Zumino-Witten model based on the affine extension of  $G$ . Notice that it is parametrized by an integer  $s \in \mathbb{Z}$ . In the second expression we have particularized for the case  $G = U(N)$ . At the level of anyon physics, the framing contribution in the Chern-Simons theory describes how anyons rotate while they wind around each other [34].

For example with the choice  $h = \sum_{n=1}^{\infty} \ln(1 - q^n)$  to account for the zeroth order Fourier coefficients of the potential, the correlator is exactly the theta function matrix

model, whose partition function is exactly [28]

$$\begin{aligned} & \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_N | e^{-\hat{H}_{CS}} | \underbrace{\downarrow, \dots, \downarrow}_N, \uparrow, \dots \rangle \\ & = e^{-i\pi N(N+1)} \prod_{i=1}^{N-1} (1 - q^i)^{N-i}. \end{aligned}$$

To account for the framing contribution (14) one can simply add the corresponding term in the magnetic field  $h$  taking into account the general result (9).

To conclude, there are several interesting open questions. First, to what extent finite spin chains can be engineered to reproduce the observables of a Chern-Simons theory with matter. This theory is no longer topological and the degeneracy at zero energy of Chern-Simons theory is broken. It would be interesting to understand if the finite size of the spin chain model achieves this in a natural way. Second, we have seen that the XX models extended with additional interactions have remarkable properties from the point of view of the theory of exactly solvable systems since its correlation functions have an immediate connection with the WZW model and admit very well-known exact expressions for every  $N$ , something that does not hold in the case of the ordinary XX model. Thus, it would be interesting now to study the properties of the Hamiltonian, including the study of spectral gap properties and the establishment of its universality class, which is expected to be the same as that of the XX model, due to the very fast decay of the additional interactions. Comparison with other exactly solvable spin chains with infinitely many interactions such as the Inozemtsev model [36] would be also of interest.

Another clear open problem is to extend this description to the case of knot and link polynomials invariants. Very recently it has been shown that a number of colored HOMFLY polynomial invariants can be written as a terminating basic hypergeometric function [37, 38]. This is a concrete realization of a known property of such polynomials: they are  $q$ -holonomic systems [39]. The key to relate this to the results presented here is the fact that these functions admit an integral representation which is precisely a unitary matrix model whose potential can be shown to be of the same type considered here. Notice that this approach is different and will not use any result on Chern-Simons theory yet essentially the same spin chain model would be the one to emerge. These results will be presented in detail elsewhere.

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- [1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phy. Rep.* **209**, (1991) 129–340.
- [2] E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).
- [3] S. C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82–85 (1989).
- [4] C. Nayak et al., *Rev. Mod. Phys.* **80**, 1083 (2008).
- [5] J. K. Pachos, *Introduction to Topological Quantum Computation*, Cambridge University Press (2012).
- [6] G. K. Brennen and J. K. Pachos, *Proceed. Roy. Soc. A* **464**, 1–24 (2008).
- [7] M. Marino, *Rev. Mod. Phys.* **77**, 675 (2005), [hep-th/0406005].
- [8] J. I. Cirac, P. Maraner, J. K. Pachos, *Phys. Rev. Lett.* **105** 190403 (2010)
- [9] K. Temme, T. J. Osborne, K. G. Vollbrecht, D. Poulin, and F. Verstraete, *Nature* **471**, 87 (2011).
- [10] S. P. Jordan, K. S. M. Lee and J. Preskill, *Science* **336**, 1130 (2012).
- [11] M. Lewenstein et al., *Adv. Phys.* **56**, 243 (2007).
- [12] M. Lewenstein, A. Sanpera and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating quantum many-body systems*, Oxford University Press, (2012).
- [13] L. Tagliacozzo et al., *Ann. Phys.* **330**, 160 (2013).
- [14] D. Banerjee et al., *Phys. Rev. Lett.* **110**, 125303 (2013).
- [15] E. Zohar, J. I. Cirac and B. Reznik, *Phys. Rev. Lett.* **110**, 055302 (2013); *Ibid.* 125304.
- [16] D. Pérez-García and M. Tierz, arXiv:1305.3877.
- [17] R. J. Szabo and M. Tierz, *Nucl. Phys. B* **876**, 234 (2013) [arXiv:1305.1580 [hep-th]].
- [18] M. Tierz, *Mod. Phys. Lett. A* **19**, 1365 (2004) [hep-th/0212128].
- [19] V. Jones, *Ann. of Math.* **126**, 335 (1987)
- [20] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Springer-Verlag (1997).
- [21] W.B.R. Lickorish, *An Introduction to Knot Theory*, Springer-Verlag (1997)
- [22] V. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, De Gruyter Studies in Mathematics (2010)
- [23] F. A. Bais, J. C. Romers, *New J. Phys.* **14**, 035024 (2012)
- [24] N. M. Bogoliubov, A.G. Pronko and J. Timonen, *Zap. Nauchn. Sem. POMI*, **403**, 5–18 (2012), arXiv:1102.5639.
- [25] N.M. Bogoliubov and C. Malyshev, *St. Petersburg Math. Jour.* **22**, 359 (2011).
- [26] D. Bump, P. Diaconis, *J. Comb. Theor. A* **97**, 252 (2002).
- [27] T. Okuda, *J. High Energy Phys.* **0503** (2005) 047 [arXiv:hep-th/0409270].
- [28] M. Romo and M. Tierz, *Phys. Rev. D* **86** (2012) 045027 [arXiv:1103.2421 [hep-th]].
- [29] R. J. Szabo and M. Tierz, *J. Phys. A* **43** (2010) 265401 [arXiv:1003.1228 [hep-th]].
- [30] J. Baik and Z. Liu, *Int. Math. Res. Not.* (2013) arxiv:1212.4467
- [31] S. Jain, S. Minwalla, T. Sharma, T. Takimi, S. R. Wadia and S. Yokoyama, [arXiv:1301.6169 [hep-th]]; T. Takimi, *JHEP* **07**, 177, (2013), [arXiv:1304.3725 [hep-th]].
- [32] R. J. Szabo and M. Tierz, *J. Math. Phys.* **53** (2012) 103502 [arXiv:1005.5643 [hep-th]].
- [33] M. F. Atiyah, “On framings of 3-manifolds,” *Topology* **29**, 1 (1990).
- [34] M. Freedman et al., *Annals of Physics* **310** (2004) 428–492.
- [35] C. Tracy and H. Widom, *SIAM J. Matrix Anal. Appl.* **23** (2002), 1194–1196.
- [36] V.I. Inozemtsev, *Lett. Math. Phys.* **17** (1989) 11
- [37] G. Giasemidis and M. Tierz, arXiv:1401.8171 [hep-th].
- [38] To appear
- [39] S. Garoufalidis and T.T.Q. Le, *Geom. Topol.* **9** (2005) 1253–1293
- [40] The discussion on experimental accessibility made in [16] for the XX-model applies also here